Forward-Backward Induction

We have the same 2 steps in a regular proof by induction, the initial case and the inductive step, but the inductive step is composed by two different steps.

- Initial case: $P(0)$
- Inductive step:
  - Forward step: $P(k) \to P(f(k))$ where f is an increasing function.
  - Backward step: $P(k) \to P(k-1)$

Quest: More patterns in forward backward induction

Find an application of a proof that requires more than one forward and/or backward step. Example:

- $f_1(k) = 2^k$
- $f_2(k) = 2^k - 1$
- $b(k) = k - 2$

Example brilliant.org: AM-GM Inequality

Initial Case: $\frac{a+b}{2} \geq \sqrt{ab}$

$$
\begin{aligned}
(a-b)^2 \geq 0 \leftrightarrow \\
& a^2 - 2ab + b^2 \geq 0 \leftrightarrow \\
& a^2 + 2ab + b^2 - 4ab \geq 0 \leftrightarrow \\
& a^2 + 2ab + b^2 \geq 4ab \leftrightarrow \\
& a^2 + 2ab + b^2 \geq 4ab \leftrightarrow \\
& (a+b)^2 \geq 4ab \leftrightarrow \\
& |a+b| \geq 2\sqrt{ab} \leftrightarrow && \text{Since both a and b are positive so is their addition} \\
& \to a+b \geq 2\sqrt{ab} \\
& \to \frac{a+b}{2} \geq \sqrt{ab}
\end{aligned}
$$

Inductive Step:

Induction Hypothesis:

$$\frac{\sum_{i=1}^{k} a_i}{k} \geq \sqrt[k]{\prod_{i=1}^{k} a_i}$$

Forward pass: We want to show,

$$\frac{\sum_{i=1}^{2k} a_i}{2k} \geq \sqrt[2k]{\prod_{i=1}^{2k} a_i}$$

We can start by splitting the summation in the left hand side,

$$\begin{aligned} a_1 + a_2 + \cdots + a_{2k} &= \frac{\frac{a_1+a_2+\cdots+a_k}{k} + \frac{a_{k+1}+a_{k+2}+\cdots+a_{2k}}{k}}{2} \\ &\geq \frac{\sqrt[k]{a_1a_2\cdots a_k} + \sqrt[k]{a_{k+1}a_{k+2}\cdots a_{2k}}}{2} \\ &\geq \sqrt{\sqrt[k]{a_1a_2\cdots a_k}\sqrt[k]{a_{k+1}a_{k+2}\cdots a_{2k}}} \\ &= \sqrt[2k]{a_1a_2\cdots a_{2k}} \\ &\square \end{aligned}$$

This completes the proof for the forward pass.

Backward pass: We want to show,

$$\frac{\sum_{i=1}^{k-1} a_i}{k-1} \geq \sqrt[k-1]{\prod_{i=1}^{k-1} a_i}$$

We start by using the inductive hypothesis,

$$\begin{aligned} \frac{a_1 + a_2 + \cdots + a_k}{k} &\geq \sqrt[k]{a_1a_2\cdots a_k} \\ \frac{a_1 + a_2 + \cdots + a_{k-1} + \frac{a_1+a_2+\cdots+a_{k-1}}{k-1}}{k} &\geq \sqrt[k]{a_1a_2\cdots a_{k-1}\cdot\frac{a_1+a_2+\cdots+a_{k-1}}{k-1}} \qquad (*1) \\ \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1} &\geq \sqrt[k]{a_1a_2\cdot\cdot a_{k-1}\cdot\frac{a_1+a_2+\cdots+a_{k-1}}{k-1}} \\ \left(\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}\right)^k &\geq a_1a_2\cdots a_{k-1}\cdot\frac{a_1+a_2+\cdots+a_{k-1}}{k-1} \\ \left(\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}\right)^{k-1} &\geq a_1a_2\cdots a_{k-1} \\ \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1} &\geq \sqrt[k-1]{a_1a_2\cdots a_{k-1}} \\ &\square \end{aligned}$$

This completes the backward pass and the proof of the AM-GM Inequality. (*1) The used property is: the arithmetic mean of k numbers in which one of them is the mean of the other $k-1$ numbers is actually the mean of the same $k-1$ numbers.

Example 3 Intro to Uni Math, Sheet 1: Every integer has an unique expansion in base b

Show that every integer $x \geq 1$ and for a base $b \geq 2$ has an unique expansion with the following form,

$$x = a_0b^0 + a_1b^1 + a_2b^2 + \ldots$$

We start by proving the existence of an expansion for every integer $x \geq 1$ by forward-backward induction.

Existence The initial case consists of proving the existence of a representation for $x = 1$.

$$x = 1b^0$$

$$\square$$

The inductive step is divided into two different proofs, one for the forward pass and another for the backward one.

Forward pass: $P(k) \to P(b^k)$

$$b^k = b^k + \sum_{i=0}^{k-1} 0b^i$$

Backward pass: $P(k) \to P(k-1)$

The coefficients of the predecessor $k-1$ are noted as c_i, where j is the index of the first non-zero coefficient.

$$c_i = \begin{cases} b-1 & , \ i < j \\ c_i - 1 & , \ i = j \\ c_i & , \ i > j \end{cases}$$

This concludes the proof of the backward pass and the proof of existence of an expansion for every integer in base b.

Uniqueness This part of the proposition is proved by contradiction where we show that for the minimum integer that has more than one expansion in base b, that we can build smaller integers that also have more than one expansion.

Lets assume that this integer is w and that the two different representations are the following,

$$\begin{aligned} w &= (a_0, a_1, \ldots, a_i, \ldots, a_n) \\ &= (c_0, c_1, \ldots, c_i, \ldots, c_n) \end{aligned}$$

Let i be the index of the first different entry such that $a_i \neq c_i$. We can also assume that if $i \neq 0$ then every other coefficient must be zero because if it weren't then we could build a smaller integer that also had two different representations. If we can make this assumption, then we can try to describe the coefficients of the predecessor $w-1$. Lets assume that $c_i - a_i > 0$ and that $a_i \neq 0$

$$\begin{aligned} w - 1 &= (b-1, b-1, \ldots, a_i - 1, \ldots, a_n) \\ &= (b-1, b-1, \ldots, c_i - 1, \ldots, c_n) \end{aligned}$$

We can set the coefficients before i to zero so that we have a smaller integer with more than one expansion in base b. In the case that $a_i = 0$, the ith position would have $b-1$ in the first representation and since $c_i - 1$ is upper bounded by $b-2$, we can still build a smaller integer that has two different representations. In the case of $i = 0$ then the first coefficient of the predecessor $w-1$ is going to have at least two different expansions since the first representation would have either $a_i - 1$ and the second $c_i - 1$ (and they are different) or the first would have $b-1$ and the second would be upper bounded by $b-2$.

[Really confusing this last part of the proof! Ask for help!]