Double Induction

We start by defining the addition of two natural numbers recursively,

Definition: Recursive Definition of Addition of Two Natural Numbers

Let m and n be natural numbers.

- m+0=m
- m+(n+1)=(m+n)+1

Example Intro to Uni Math ex.2: Commutativity of Addition of Natural Numbers

We want to show that the addition is commutative using its recursive definition,

$$m + n = n + m$$

Let $n = 0$. We want to show that $m + 0 = 0 + m$. We show by induction on m. The base case is skipped. We assume $k + 0 = 0 + k$. We want to prove $0 + (k + 1) = (k + 1) + 0$

$$\begin{aligned} 0 + (k+1) &= (0+k)+1 && \text{Recursive Part of Definition} \\ &= (k+0)+1 && \text{Inductive Hypothesis} \\ &= k+1 && \text{Base Case for Recursive Definition LR} \\ &= (k+1)+0 && \text{Base Case for Recursive Definition RL} \end{aligned}$$

Let $n = 1$. We want to show that $m + 1 = 1 + m$. We show by induction on m. Base case $0 + 1 = 1 + 0$ was already proven when $n = 0 \wedge m = 1$. We assume $k + 1 = 1 + k$. We want to prove $1 + (k + 1) = (k + 1) + 1$

$$\begin{aligned} (k+1)+1 &= (1+k)+1 && \text{Inductive Hypothesis} \\ &= 1+(k+1) && \text{Recursive Part of Definition RL} \end{aligned}$$

□

We want to show that $m + n = n + m$. We prove by induction on n. The base case was already shown. We assume $m + k = k + m$. We want to prove $m + (k + 1) = (k + 1) + m$

$$\begin{aligned} (m+k)+1 &= (k+m)+1 && \text{Inductive Hypothesis} \\ &= 1+(k+m) && \text{Using (ii)} \\ &= (1+k)+m && \text{Recursive Part of Definition} \\ &= (k+1)+m && \text{Using (ii)} \end{aligned}$$

□